



Computability of the Metric Projection Onto Finite-dimensional Linear Subspaces

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Abstract

We show that given a computable Banach space X and a finite-dimensional subspace U of X the set of elements of best approximation of $x \in X$ (by elements of U) can be computed as a compact set with negative information. If X is uniformly convex, we can even compute the (unique) element of best approximation. Furthermore, given a uniformly convex computable Banach space X the mapping $U \mapsto P_U$ that maps each finite dimensional linear subspace to the corresponding (single-valued) metric projection is computable.

Keywords: Computable functional analysis, best approximation, metric projection.

1 Introduction

In approximation theory the problem of best approximation can also be studied using the methods of functional analysis [10]. In this paper we will study some parts of the problem of best approximation in Banach spaces by elements of closed convex sets and by elements of finite-dimensional linear subspaces from the view of computable analysis using the representation based approach of Weihrauch [11].

Given a normed space X the problem of best approximation is the problem of finding, for a given subset $G \subseteq X$ and a point $x \in X$, a point $g_0 \in G$ that is the

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nearest point to x among all elements of G , that is

$$\|x - g_0\| = \inf_{g \in G} \|x - g\| = \text{dist}(x, G) \quad (1)$$

where $\text{dist}(x, G)$ denotes the distance between a point $x \in X$ and a subset $G \subseteq X$. In general the existence of such an element g_0 is not guaranteed nor has it to be unique since every element of G with the property of equation (1) is an *element of best approximation* of x in G . Therefore, by $\mathcal{P}_G(x)$ we denote the set of all elements of best approximation of an element $x \in X$ in a subset $G \subseteq X$, that is

$$\mathcal{P}_G(x) := \{g_0 \in G \mid \|x - g_0\| = \inf_{g \in G} \|x - g\|\} = \{g_0 \in G \mid \|x - g_0\| = \text{dist}(x, G)\}.$$

Then g_0 is an element of best approximation of x in G if and only if $g_0 \in \mathcal{P}_G(x)$. For a given subset G and an element x , the set $\mathcal{P}_G(x)$ may be empty or have got more than one element. If X is a uniformly convex Banach space and G a closed convex subset of X then $\mathcal{P}_G(x)$ consists of exactly one element for all $x \in X$. In this case a single-valued total function $P_G : X \rightarrow X$ can be defined by $P_G(x) := g_0 : \iff \mathcal{P}_G(x) = \{g_0\}$. P_G maps each element of X to its uniquely defined best approximation in G . P_G is called the *metric projection* onto G .

In this paper we will present some conditions under that the mappings $G \rightarrow \mathcal{P}_G$ and $G \rightarrow P_G$ become computable in some sense defined later.

The problem of best approximation has also extensively been studied in constructive analysis. We mention, in particular, the results in [6], [7], [1, pp. 309–313], and [5, pp. 88–92]. It seems that our Theorem 4.5(i) and Corollary 6.1 could also be derived from those results via realizability theory. However, we are not aware of any counterpart of Theorem 5.2 and Theorem 6.2 in constructive analysis.

In the next section we briefly define some concepts from computable analysis that we will need to present our results. In section 3 we formulate some technical results about finite linear combinations and linear independence that we use in the following sections. In section 4 we define a representation of finite-dimensional subspaces that is used to present the computability results about best approximation in finite-dimensional subspaces. Furthermore, we compare our representation with further representations for finite-dimensional subspaces that can be derived from known representations for closed subsets. In section 5 we present a first result about the computability of the metric projection in case of closed convex subsets. In section 6 we prove some better results about the computability of $\mathcal{P}_G(x)$ and $P_G(x)$ for finite-dimensional linear subspaces G , which are special closed convex subsets. In the last section we briefly summarize our result about the computability of the metric projection.

2 Computable Banach Spaces

In this section we briefly define some concepts from computable analysis. Computability on Banach spaces is extensively studied by Pour-El and Richards in their

book [9]. We will study this subject using the representation based approach to computable analysis of Weihrauch [11]. The representation based approach is essentially compatible to the sequential approach of Pour-El and Richards, but it can more flexibly be adapted to higher degrees of uniformity. We refer the reader to [11] for all concepts that are left undefined here. In the following we assume that Banach spaces are defined over the field \mathbb{F} , which might either be \mathbb{R} or \mathbb{C} .

Definition 2.1 [Computable Banach space] A *computable normed space* $(X, \|\cdot\|, e)$ is a separable normed space $(X, \|\cdot\|)$ together with a fundamental sequence $e : \mathbb{N} \rightarrow X$ (that is the linear span of $\text{range}(e)$ is dense in X) such that the induced metric space is a computable metric space. A *computable Banach space* is a computable normed space that is a Banach space, that is complete.

The induced computable metric space is the space (X, d, α_e) where d is given by $d(x, y) := \|x - y\|$ and $\alpha_e : \mathbb{N} \rightarrow X$ is defined by $\alpha_e \langle k, \langle n_0, \dots, n_k \rangle \rangle := \sum_{i=0}^k \alpha_{\mathbb{F}}(n_i) e_i$. Here $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}$ in case of $\mathbb{F} = \mathbb{R}$ and $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}[i]$ in case of $\mathbb{F} = \mathbb{C}$. We assume that there is some $n \in \mathbb{N}$ with $\alpha_{\mathbb{F}}(n) = 0$. The linear operations (vector space addition and scalar multiplication) are automatically computable for any computable normed space.

In general, a space (X, d, α) is called a *computable metric space*, if (X, d) is a metric space with a dense sequence α such that $d \circ (\alpha \times \alpha)$ is a computable double sequence. If not mentioned otherwise, then we assume that all computable Banach spaces X are represented by their Cauchy representation δ_X (of the induced metric space). The *Cauchy representation* $\delta_X : \subseteq \Sigma^\omega \rightarrow X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$, if it encodes a sequence $(\alpha(n_i))_{i \in \mathbb{N}}$, which rapidly converges to x , where rapid means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$. Here Σ^ω denotes the set of infinite sequences over some finite set Σ (the *alphabet*) and Σ^ω is endowed with the product topology with respect to the discrete topology on Σ .

In general a *representation* of a set X is a surjective map $\delta : \subseteq \Sigma^\omega \rightarrow X$. Here the inclusion symbol “ \subseteq ” indicates that the corresponding map might be partial. Given representations $\delta : \subseteq \Sigma^\omega \rightarrow X$ and $\delta' : \subseteq \Sigma^\omega \rightarrow Y$, a map $f : \subseteq X \rightarrow Y$ is called (δ, δ') -*computable*, if there exists a computable map $F : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ such that $\delta' F(p) = f \delta(p)$ for all $p \in \text{dom}(f \delta)$. Analogously, one can define computability for multi-valued functions $f : \subseteq X \rightrightarrows Y$. In this case the equation above has to be replaced by the condition $\delta' F(p) \in f \delta(p)$. Here a function $F : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ is called *computable* if there exists a Turing machine which computes F . Similarly, one can define the concept of *continuity* with respect to representations, where the computable function F is replaced by a continuous function.

Cauchy representations of computable metric spaces X are known to be *admissible* and for such representations continuity with respect to representations coincides with ordinary continuity. If X, Y are computable metric spaces, then we assume that the space $\mathcal{C}(X, Y)$ of continuous functions $f : X \rightarrow Y$ is represented by $[\delta_X \rightarrow \delta_Y]$, which is a canonical function space representation. This representation satisfies two characteristic properties, evaluation and type conversion, which can be performed

computably (see [11] for details). If $Y = \mathbb{F}$, then we write for short $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{F})$.

We say that a representation δ is *computably reducible* to another representation δ' of the same set, in symbols $\delta \leq \delta'$, if there is a computable function $F : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ such that $\delta(p) = \delta'(F(p))$ for all $p \in \text{dom}(\delta)$. This is equivalent to the fact that the identity $\text{id} : X \rightarrow X$ is (δ, δ') -computable. Two representations are said to be *computably equivalent*, if they are mutually computably reducible to each other, in symbols $\delta \equiv \delta'$.

In the following we have to deal with closed and compact subsets of normed and metric spaces. Given a metric space X we denote the set of all closed subsets of X by $\mathcal{A}(X)$ and the set of all compact subsets of X by $\mathcal{K}(X)$. To represent these spaces we use the representations for closed and compact subsets of metric spaces that are defined and studied by Brattka and Presser in [4].

For closed subsets, we will use the representations $\delta_{\text{dist}}^>$, $\delta_{\text{dist}}^<$ and $\delta_{\text{dist}}^\equiv$, which represent closed subsets by their distance functions with negative, positive, and full information, as well as the representations δ_{range} , which represents a closed subset by a dense sequence, and δ_{fiber} , which represents a closed subset by a total function such that the set is the preimage of $\{0\}$. For compact subsets, we will use the representations δ_{cover} and $\delta_{\text{min-cover}}$, which represent a compact subset by all finite “rational” covers of the set and by all minimal finite “rational” covers of the set, respectively. For further information about these representations we refer the reader to [4].

If X and Y are normed spaces, we denote the set of all bounded linear operators from X to Y by $\mathcal{B}(X, Y)$ and assume $\mathcal{B}(X, Y)$ is represented as subset of $\mathcal{C}(X, Y)$ by the restriction of $[\delta_X \rightarrow \delta_Y]$ to $\mathcal{B}(X, Y)$. By $S_X(a, r)$ we denote the sphere in X with center $a \in X$ and radius $r \geq 0$. By $B_X(a, r)$ we denote the corresponding closed ball. In case of $a = 0$ and $r = 1$, we denote the unit sphere and closed unit ball by S_X and B_X , respectively.

3 Linear Combinations and Linear Independence

In this section we summarize some results about finite linear combinations and linear independence that we need in the following sections. To formulate our results, we first introduce two representations for finite tuples and functions on finite tuples.

Given a represented³ space (X, δ) , we denote the set of all finite tuples (x_1, \dots, x_k) for some $k \in \mathbb{N}$ and $x_1, \dots, x_k \in X$ by $X^* = \bigcup_{n \in \mathbb{N}} X^k$. We equip X^* with the canonical representation δ^* that represents an element (x_1, \dots, x_k) of X^* by the number k of elements and a name of the tuple as an element of X^k .

Definition 3.1 [Finite tuples] Let (X, δ) be a represented space. By

$$X^* := \bigcup_{k \in \mathbb{N}} X^k$$

³ In the following this will usually be a metric or normed space with the corresponding Cauchy representation.

we denote the set of all finite tuples over X . We equip X^* with the representation $\delta^* : \subseteq \Sigma^\omega \rightarrow X^*$ defined by

$$\delta^* \langle p, q \rangle = \bar{x} : \iff \delta_{\mathbb{N}}(p) = k, \bar{x} \in X^k \text{ and } \delta^k(q) = \bar{x}$$

where δ^k is the standard representation of the product space X^k .

If not mentioned otherwise, X^k and X^* are equipped with δ^k and δ^* respectively. Given another represented space (Y, δ') , we denote the set of all continuous functions $f : X^k \rightarrow Y$ for some $k \in \mathbb{N}$ by $\mathcal{C}^*(X, Y) := \bigcup_{k \in \mathbb{N}} \mathcal{C}(X^k, Y)$. We equip $\mathcal{C}^*(X, Y)$ with the canonical representation δ_{\rightarrow}^* that represents an element $f : X^k \rightarrow Y$ of $\mathcal{C}^*(X, Y)$ by the dimension k of the source space X and a name of the function as element of $\mathcal{C}(X^k, Y)$.

Definition 3.2 Let (X, δ) and (Y, δ') be represented spaces. By

$$\mathcal{C}^*(X, Y) := \bigcup_{k \in \mathbb{N}} \mathcal{C}(X^k, Y)$$

we denote the set of all continuous functions $f : X^k \rightarrow Y$ for some $k \in \mathbb{N}$. We equip $\mathcal{C}^*(X, Y)$ with the representation δ_{\rightarrow}^* defined by

$$\delta_{\rightarrow}^* \langle p, q \rangle = f : \iff \delta_{\mathbb{N}}(p) = k, f \in \mathcal{C}(X^k, Y) \text{ and } [\delta^k \rightarrow \delta'](q) = f.$$

If not mentioned otherwise, $\mathcal{C}(X^k, Y)$ and $\mathcal{C}^*(X, Y)$ are equipped with $[\delta^k \rightarrow \delta']$, the standard representation of the function space $\mathcal{C}(X^k, Y)$, and δ_{\rightarrow}^* , respectively.

Now we are prepared to formulate the above mentioned computability results about finite linear combinations and linear independence. Given a computable normed space X , finite linear combinations are computable in the following uniform way. By $\mathcal{B}(\mathbb{F}^k, X)$ we denote the set of all bounded linear operators from \mathbb{F}^k to X .

Proposition 3.3 *Let X be a computable normed space. We define a mapping $\text{LC} : X^* \rightarrow \mathcal{C}^*(\mathbb{F}, X)$ by*

$$\text{LC}(\bar{x}) \in \mathcal{C}(\mathbb{F}^k, X) : \iff \bar{x} \in X^k$$

and

$$\text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k) := \sum_{i=1}^k \alpha_i x_k$$

for all $(x_1, \dots, x_k) \in X^*$ and $\alpha_1, \dots, \alpha_k \in \mathbb{F}$.

- (i) LC is $(\delta_X^*, \delta_{\rightarrow}^*)$ -computable.
- (ii) $\text{LC}(x_1, \dots, x_k) \in \mathcal{B}(\mathbb{F}^k, X)$ for all $(x_1, \dots, x_k) \in X^*$.

Proof. The claims directly follow from the definition of LC and the fact that X is a computable normed space. \square

Given a normed space X , by IND_X we denote the set

$$\text{IND}_X := \{(x_1, \dots, x_k) \in X^* \mid (x_1, \dots, x_k) \text{ is linearly independent}\}$$

of all finite tuples that consist of linearly independent elements of X .⁴ IND_X is a r.e. open subset of X^* (see also [9, Effective Independence Lemma] and [13, Lemma 10]).

Proposition 3.4 *Let X be a computable normed space.*

(i) *There exists a $(\delta_X^*, \delta_{\mathbb{R}})$ -computable function $F : X^* \rightarrow \mathbb{R}$ such that*

$$F(x_1, \dots, x_k) = 0 \iff (x_1, \dots, x_k) \text{ is not linearly independent}$$

or equivalently

$$F^{-1}[\{0\}] = X^* \setminus \text{IND}_X .$$

(ii) IND_X is an r.e. open subset of X^* .

In the proof of Proposition 3.4, we use the following technical lemma, which we will also use in the proof of Proposition 3.6.

Lemma 3.5 *Let X be a computable normed space. We define mappings $F_1 : X^* \rightarrow \mathbb{R}$ and $F_2 : X^* \rightarrow \mathbb{R}$ by*

$$F_1(x_1, \dots, x_k) := \max \{ \| \text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k) \| \mid \|(\alpha_1, \dots, \alpha_k)\| = 1 \}$$

and

$$F_2(x_1, \dots, x_k) := \min \{ \| \text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k) \| \mid \|(\alpha_1, \dots, \alpha_k)\| = 1 \}$$

for all $(x_1, \dots, x_k) \in X^$. Then F_1 and F_2 are $(\delta_X^*, \delta_{\mathbb{R}})$ -computable.*

Proof. The compact unit sphere $S_k := \{(\alpha_1, \dots, \alpha_k) \in \mathbb{F}^k \mid \|(\alpha_1, \dots, \alpha_k)\| = 1\}$ of \mathbb{F}^k is computable as a compact set for all $k \in \mathbb{N}$. Since the image of a compact set under a continuous function and the minimum and maximum of a compact set of real numbers are computable [12,11] the computability of F_1 and F_2 follows. \square

Proof. [Proof of Proposition 3.4]

(i) We define $F := F_2$ where $F_2 : X^* \rightarrow \mathbb{R}$ is the $(\delta_X^*, \delta_{\mathbb{R}})$ -computable function of Lemma 3.5. Then we have

$$(x_1, \dots, x_k) \text{ is not linearly independent} \iff F(x_1, \dots, x_k) = 0$$

for all $(x_1, \dots, x_k) \in X^*$. It follows

$$(x_1, \dots, x_k) \in F^{-1}[\{0\}] \iff (x_1, \dots, x_k) \in X^* \setminus \text{IND}_X .$$

⁴ We call a tuple $(x_1, \dots, x_k) \in X^*$ linearly independent if $\sum_{i=1}^k \alpha_i x_i = 0$ for $\alpha_1, \dots, \alpha_k \in \mathbb{F}$ implies $\alpha_i = 0$ for all $i = 1, \dots, k$.

- (ii) As we have shown in the previous item, $X^* \setminus \text{IND}_X$ is co-r. e. closed in X^* . It follows that IND_X is r. e. open in X^*

□

Given a computable normed space X , the function $\text{LC}(\bar{x})$ is injective if \bar{x} consists of linearly independent elements of X , that is if $\bar{x} \in \text{IND}_X$. In this case, the partial inverse of $\text{LC}(\bar{x})$ exists and is computable. To prove this result, we first prove that the norm of $\text{LC}(\bar{x})$ is computable in a uniform way and that the norm of the partial inverse $(\text{LC}(\bar{x}))^{-1}$ is computable in the same way if $\bar{x} \in \text{IND}_X$.

Proposition 3.6 *Let X be a computable normed space.*

- (i) *The mapping*

$$N : X^* \rightarrow \mathbb{R}, N(\bar{x}) := \|\text{LC}(\bar{x})\|$$

is $(\delta_X^, \delta_{\mathbb{R}})$ -computable.*

- (ii) *The mapping*

$$N_{\text{inv}} : \subseteq X^* \rightarrow \mathbb{R}, N_{\text{inv}}(\bar{x}) := \|(\text{LC}(\bar{x}))^{-1}\|$$

with $\text{dom}(N_{\text{inv}}) := \text{IND}_X$ is $(\delta_X^, \delta_{\mathbb{R}})$ -computable.*

Proof.

- (i) We have $N = F_1$ where $F_1 : X^* \rightarrow \mathbb{R}$ is the $(\delta_X^*, \delta_{\mathbb{R}})$ -computable function of Lemma 3.5.
- (ii) Given $(x_1, \dots, x_k) \in \text{dom}(N_{\text{inv}}) = \text{IND}_X$ we have $F_2(x_1, \dots, x_k) > 0$ where $F_2 : X^* \rightarrow \mathbb{R}$ is the $(\delta_X^*, \delta_{\mathbb{R}})$ -computable function of Lemma 3.5. Furthermore, in this case the inverse operator $(\text{LC}(x_1, \dots, x_k))^{-1}$ exists as a linear bounded operator and we have

$$\begin{aligned} & \|(\text{LC}(x_1, \dots, x_k))^{-1}\| \\ &= (\min \{ \|\text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k)\| \mid \|(\alpha_1, \dots, \alpha_k)\| = 1 \})^{-1} \\ &= (F_2(x_1, \dots, x_k))^{-1}. \end{aligned}$$

It follows $N_{\text{inv}}(x_1, \dots, x_k) = (F_2(x_1, \dots, x_k))^{-1}$ if $(x_1, \dots, x_k) \in \text{dom}(N_{\text{inv}})$, and $N_{\text{inv}}(x_1, \dots, x_k)$ and $(F_2(x_1, \dots, x_k))^{-1}$ are undefined if $(x_1, \dots, x_k) \notin \text{dom}(N_{\text{inv}})$. We obtain $N_{\text{inv}} = F_2$ and N_{inv} is $(\delta_X^*, \delta_{\mathbb{R}})$ -computable.

□

Proposition 3.7 *Let X be a computable normed space. We define a mapping*

$$\text{LC}'_{\text{inv}} : \subseteq X^* \times X \rightarrow \mathbb{F}^*, \text{LC}'_{\text{inv}}(\bar{x}, y) := (\text{LC}(\bar{x}))^{-1}(y)$$

with $\text{dom}(\text{LC}'_{\text{inv}}) := \{(\bar{x}, y) \in X^ \times X \mid \bar{x} \in \text{IND}_X \text{ and } y \in \text{range}(\text{LC}(\bar{x}))\}$*

- (i) $\text{LC}'_{\text{inv}}(\bar{x}, y) \in \mathbb{F}^k$ for all $(\bar{x}, y) \in \text{dom}(\text{LC}'_{\text{inv}})$ with $\bar{x} \in X^k$.
- (ii) LC'_{inv} is $(\delta_X^*, \delta_X, \delta_{\mathbb{F}}^*)$ -computable.

Proof.

- (i) This claim directly follows from the definition of LC'_{inv} .

- (ii) Given $\bar{x} = (x_1, \dots, x_k) \in \text{IND}_X$, we can compute $\text{LC}(\bar{x})$ and the norm $\|(\text{LC}(\bar{x}))^{-1}\| = N_{\text{inv}}(\bar{x})$. Furthermore, $\text{LC}(\bar{x})$ is injective in this case and we have $\text{range}(\text{LC}(\bar{x})) = \text{span}\{x_1, \dots, x_k\}$. Let $L \in \mathbb{N}$ such that $2^L \geq \|(\text{LC}(\bar{x}))^{-1}\|$. We can effectively find such an L . Given additionally $y \in X$, for each $k \in \mathbb{N}$ we can effectively search some $(\alpha_1, \dots, \alpha_k) \in \mathbb{F}^*$, such that $\|\text{LC}(\bar{x})(\alpha_1, \dots, \alpha_k) - y\| \leq 2^{-k-L}$. If $y \in \text{range}(\text{LC}(\bar{x}))$ we will find such a tuple $(\alpha_1, \dots, \alpha_k)$ at some time, otherwise $(\bar{x}, y) \notin \text{dom}(\text{LC}'_{\text{inv}})$ and our search procedure will never stop. In the first case, we have

$$\begin{aligned} \|\text{LC}'_{\text{inv}}(\bar{x}, y) - (\alpha_1, \dots, \alpha_k)\| \\ \leq \|(\text{LC}(\bar{x}))^{-1}\| \cdot \|y - \text{LC}(\bar{x})(\alpha_1, \dots, \alpha_k)\| \\ \leq 2^{-k}. \end{aligned}$$

Hence given $(\bar{x}, y) \in \text{dom}(\text{LC}'_{\text{inv}})$ we can compute a sequence in \mathbb{F}^* that converges fast to $\text{LC}'_{\text{inv}}(\bar{x}, y)$.

□

4 Finite-dimensional Linear Subspaces

Given a normed space X , we denote the set of all k -dimensional linear subspaces of X by $\mathcal{L}_X^{(k)}$, the set of all finite-dimensional linear subspaces by $\mathcal{L}_X^{(*)}$, and the set of all linear subspace by \mathcal{L}_X . Obviously, we have $\mathcal{L}_X^{(*)} = \bigcup_{k \in \mathbb{N}} \mathcal{L}_X^{(k)}$, and $\mathcal{L}_X = \mathcal{L}_X^{(*)}$ holds if and only if X has finite dimension. If X is a computable normed space with standard representation δ_X , we equip X^* with the canonical representation δ^* that represents an element (x_1, \dots, x_k) of X^* by the number k of elements and a name of the tuple as an element of X^k . If not mentioned otherwise, X^k and X^* are equipped with δ^k , the standard representation of the product space X^k , and δ^* , respectively.

Since every finite-dimensional linear subspace of a normed space X is a closed subset of X , that is $\mathcal{L}_X^{(*)} \subseteq \mathcal{A}(X)$, we can use the representations for closed subsets of metric spaces that are defined in [4] as representations for $\mathcal{L}_X^{(*)}$. Additionally, we define the following representation for $\mathcal{L}_X^{(*)}$, which represents finite-dimensional linear subspaces by a basis of the subspace.

Definition 4.1 [Basis representation] Let X be a computable Banach space. We define a representation δ_{basis} of $\mathcal{L}_X^{(*)}$ by

$$\delta_{\text{basis}}(p) = U : \iff \dim(U) = k, \delta_X^*(p) = (x_1, \dots, x_k) \in \text{IND}_X \text{ and } \text{span}\{x_1, \dots, x_k\} = U$$

of the set $\mathcal{L}_X^{(*)}$ of all finite-dimensional linear subspaces of X .

Since a δ_{basis} -name of a finite-dimensional linear subspace U encodes a basis of U , we can compute the dimension and a basis of U from such a name.

Lemma 4.2 *Let X be a computable normed space.*

(i) *The mapping*

$$\dim : \mathcal{L}_X^{(*)} \rightarrow \mathbb{N}, U \mapsto \dim(U)$$

that maps each finite-dimensional subspace to its dimension is $(\delta_{\text{basis}}, \delta_{\mathbb{N}})$ -computable.

(ii) *The multi-valued mapping*

$$\text{basis} : \mathcal{L}_X^{(*)} \rightrightarrows X^*$$

defined by

$$\bar{x} = (x_1, \dots, x_k) \in \text{basis}(U) : \iff \dim(U) = k \text{ and } \{x_1, \dots, x_k\} \text{ is a basis of } U$$

is $(\delta_{\text{basis}}, \delta_X^*)$ -computable.

(iii) *The single-valued right inverse*

$$\text{basis}^{-1} : \subseteq X^* \rightarrow \mathcal{L}_X^{(*)}, (x_1, \dots, x_k) \mapsto \text{span}\{x_1, \dots, x_k\}$$

with $\text{dom}(\text{basis}^{-1}) = \text{IND}_X$ of basis is $(\delta_X^*, \delta_{\text{basis}})$ -computable.

Using δ_{basis} as representation, we can computably embed $\mathcal{L}_X^{(*)}$ into $\mathcal{A}(X)$. By $\mathcal{A}(X)$ we denote the set of all closed subsets of X and equip $\mathcal{A}(X)$ with the representations δ_{dist}^- defined in [4]. δ_{dist}^- represents a closed set by its distance function.

Proposition 4.3 *Let X be a computable normed space. The embedding*

$$\text{in} : \mathcal{L}_X^{(*)} \hookrightarrow \mathcal{A}(X), U \mapsto U$$

of $\mathcal{L}_X^{(*)}$ into $\mathcal{A}(X)$ is $(\delta_{\text{basis}}, \delta_{\text{dist}}^-)$ -computable.

Proof. Given a δ_{basis} -name of a finite-dimensional linear subspace U , we can compute a basis $\{x_1, \dots, x_k\}$ of U . It remains to show that given a basis $\{x_1, \dots, x_k\}$, we can compute the distance function of the linear subspace $U = \text{span}\{x_1, \dots, x_k\}$ as a closed subset of X . For $x \in X$ we know that $\text{dist}(x, U) \leq \|x\|$ and that there exists some $z \in U$ with $\text{dist}(x, U) = \|x - z\|$.⁵ Thus there exists some $z \in U$ with $\text{dist}(x, U) = \|x - z\|$ and $\|z\| \leq 2\|x\|$. By Proposition 3.6(ii) given linearly independent $x_1, \dots, x_k \in X$, we can compute $\alpha := \|(\text{LC}(x_1, \dots, x_k))^{-1}\|$. It holds

$$\sqrt{\sum_{i=1}^k \alpha_i^2} = \|(\alpha_1, \dots, \alpha_k)\| \leq \|(\text{LC}(x_1, \dots, x_k))^{-1}\| \cdot \left\| \sum_{i=1}^k \alpha_i x_i \right\| = \alpha \|y\|$$

for all $y = \sum_{i=1}^k \alpha_i x_i \in U$. As $z \in U \cap B_X(0, 2\|x\|)$, there exists $(\alpha_1, \dots, \alpha_k) \in \mathbb{F}^k$ with $z = \sum_{i=1}^k \alpha_i x_i$ and $\|(\alpha_1, \dots, \alpha_k)\| \leq \alpha \cdot \|z\| \leq 2\alpha\|x\|$. Thus there exists

⁵ Finite-dimensional subspaces of normed spaces are always proximal, that is every element of the normed space has a proximum in the subspace [10].

some $(\alpha_1, \dots, \alpha_k) \in B_{\mathbb{F}^k}(0, 2\alpha\|x\|)$ with $z = \text{LC}(x_1, \dots, x_k)(\alpha_1, \dots, \alpha_k)$ or equivalently $z \in \text{LC}(x_1, \dots, x_k)[B_{\mathbb{F}^k}(0, 2\alpha\|x\|)]$. Given $(x_1, \dots, x_k) \in \text{IND}_X$ and x we can compute the closed ball $B_{\mathbb{F}^k}(0, 2\alpha\|x\|)$ with center 0 and radius $2\alpha\|x\|$ as a compact subset of \mathbb{F}^k and thus also $V := \text{LC}(x_1, \dots, x_k)[B_{\mathbb{F}^k}(0, 2\alpha\|x\|)]$ as a compact set with full information. Since we have $B_X(0, 2\|x\|) \cap U \subseteq V \subseteq U$ it follows $\text{dist}(x, U) = \|z - x\| = \text{dist}(x, V)$. $\text{dist}(x, V)$ can be computed because we have got V as a compact set with full information. Hence given $(x_1, \dots, x_k) \in \text{IND}_X$ and $x \in X$ we can compute $\text{dist}(x, \text{span}\{x_1, \dots, x_k\}) = \text{dist}(x, U)$. By type conversion it follows that given $(x_1, \dots, x_k) \in \text{IND}_X$ with $U = \text{span}\{x_1, \dots, x_k\}$ we can compute the distance function of U . This is equivalent to computing a δ_{dist}^- -name of U . \square

A corresponding result in constructive analysis is Proposition 2.1 in [6].

Using some results of Brattka and Presser [4] about computable reducibility between the different representations for closed set, we immediately get the following corollary. Here by δ_{fiber} and δ_{range} we denote the representations defined in [4] that represent a closed set A by a function $f : X \rightarrow \mathbb{R}$ such that $A = f^{-1}\{0\}$ and a sequence $g : \mathbb{N} \rightarrow X$ such that $\text{range}(g)$ is dense in A , respectively.

Corollary 4.4 *Let X be a computable normed space. The embedding*

$$\text{in} : \mathcal{L}_X^{(*)} \hookrightarrow \mathcal{A}(X), U \mapsto U$$

of $\mathcal{L}_X^{()}$ into $\mathcal{A}(X)$ is $(\delta_{\text{basis}}, \delta_{\text{fiber}})$ -computable. in is also $(\delta_{\text{basis}}, \delta_{\text{range}})$ -computable if X is a Banach space.*

In the following we will compare our new representation δ_{basis} with the representations of closed subsets restricted to $\mathcal{L}_X^{(*)}$.

Theorem 4.5 *Let X be a computable normed space. Then we have*

- (i) $\delta_{\text{basis}} \leq \delta_{\text{dist}}^- | \mathcal{L}_X^{(*)}$,
- (ii) $\delta_{\text{basis}} \leq \delta_{\text{range}} | \mathcal{L}_X^{(*)}$ if X is a Banach space,
- (iii) $\delta_{\text{basis}} \leq \delta_{\text{fiber}} | \mathcal{L}_X^{(*)}$.

Proof. The given reductions follow from Proposition 4.3 and Corollary 4.4. \square

Theorem 4.5 generalizes some results in [13, Theorem 11] about representations of linear subspaces of \mathbb{R}^n .

In the following we often omit “ $| \mathcal{L}_X^{(*)}$ ” when we use the restricted representations of $\mathcal{A}(X)$ to simplify the presentation. It will always be clear if the representation of $\mathcal{A}(X)$ or the restriction to $\mathcal{L}_X^{(*)}$ is meant.

5 Metric Projections onto Convex Subsets

A subset U of a normed space X is called convex if $\lambda x + (1 - \lambda)y \in U$ holds for all $x, y \in U$ and $0 \leq \lambda \leq 1$. A normed space X is called rotund or strictly convex if

$\|tx_1 + (1-t)x_2\| < 1$ whenever $x_1, x_2 \in S_X$, $x_1 \neq x_2$, and $0 < t < 1$ [8]. Here by S_X we denote the unit sphere of X . If X is a normed space with a strictly convex norm, the *modulus of convexity* $\text{mc} : [0, 2] \rightarrow [0, 1]$ of the norm $\|\cdot\|$ is defined by

$$\text{mc}(\varepsilon) := \inf\{1 - \|\frac{1}{2}(x+y)\| \mid x, y \in S_X, \|x-y\| \geq \varepsilon\}.$$

The norm $\|\cdot\|$ is called *uniformly convex* if $\text{mc}(\varepsilon) > 0$ for $0 < \varepsilon \leq 2$.

Every finite-dimensional normed space is uniformly convex if and only if it is strictly convex [8, Prop. 5.2.14] and every uniformly convex Banach space is known to be reflexive (the Milman-Pettis theorem, see for example [8, Thm. 5.2.15]).

Given a normed space X , a subset $G \subseteq X$ of X and an element $x \in X$, we define the set

$$\mathcal{P}_G(x) := \{z \in G \mid \|x - z\| = \text{dist}_G(x)\}$$

of all elements of best approximation of x by elements of G . G is called⁶

- a *proximal set* or *set of existence* if $\mathcal{P}_G(x) \neq \emptyset$, that is $\mathcal{P}_G(x)$ contains at least one element, for all $x \in X$,
- a *semi-Chebyshev set* or *set of uniqueness* if $\mathcal{P}_G(x)$ contains at most one element for all $x \in X$,
- a *Chebyshev set* if $\mathcal{P}_G(x)$ contains exactly one element for all $x \in X$.

If a subset $G \subseteq X$ of a normed space X is a Chebyshev set we can define a total and single-valued function that maps every $x \in X$ to its uniquely defined best approximation in G .

Definition 5.1 [Metric projection] Let X be a normed space and $G \subseteq X$ a Chebyshev set. We define the *metric projection* $P_G : X \rightarrow X$ onto G by

$$y = P_G(x) : \iff \mathcal{P}_G(x) = \{y\}$$

for all $x \in X$.

The norm of a normed space X is strictly convex if and only if every nonempty (closed) convex subset of X is a set of uniqueness [8, Theorem 5.1.18]. If every nonempty closed convex subset of a normed space X is a set of existence, then X is reflexive. Additionally, a normed space X is reflexive and has got a strictly convex norm if and only if every nonempty closed convex subset is a Chebyshev set [8]. It follows that every nonempty closed convex subset of a uniformly convex Banach space X is a Chebyshev set, that is the metric projection onto it is a total and single-valued function.

If the modulus of convexity of the uniformly convex Banach space X is $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$ -computable we can even compute the metric projection of a given nonempty convex subset.

⁶ For further information about these notions we refer the reader to [10, 8].

Theorem 5.2 (Metric projection) *Let X be a computable Banach space with a uniformly convex norm and a $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$ -computable modulus of convexity mc . We define a mapping*

$$\widehat{P} : \subseteq \mathcal{A}(X) \times X \rightarrow X, (G, x) \mapsto P_G(x)$$

with

$$\text{dom}(\widehat{P}) := \{G \in \mathcal{A}(X) \mid G \neq \emptyset \text{ is convex}\} \times X$$

that maps every nonempty convex subset G of X to the corresponding metric projection. Then \widehat{P} is $(\delta_{\text{dist}}^{\text{--}}, \delta_X, \delta_X)$ -computable.

Proof. Since every nonempty closed convex set in a uniformly convex Banach space is a Chebyshev set, \widehat{P} is well-defined as a single-valued function. We have to show that given a nonempty closed convex set $G \subseteq X$, some $x \in X$, and $k \in \mathbb{N}$, we can effectively find some $z \in X$ with $\|P_G(x) - z\| < 2^{-k}$. In the following we will show even more as we can even find some $z \in G$ (and not only $z \in X$) with $\|P_G(x) - z\| < 2^{-k}$.

Since the norm of X is uniformly convex, we have $\text{mc}(\varepsilon) > 0$ for $\varepsilon > 0$, hence $\text{mc}(2^{-k}) > 0$ for all $k \in \mathbb{N}$. Furthermore, the mapping $k \mapsto \text{mc}(2^{-k})$ is $(\delta_{\mathbb{N}}, \delta_{\mathbb{R}})$ -computable. Let $G \subseteq X$ be a nonempty closed convex set and $x \in X$. Let $g_0 \in G$ be the uniquely determined best approximation of x in G , that is we have $\|x - g_0\| = \text{dist}(x, G) =: r \geq 0$. Furthermore, $r = \text{dist}(x, G) = \|x - g_0\|$ can be computed from the given information of G and x .

Let $k \in \mathbb{N}$. Then we have $r < 2^{-k-2}$ or $r > 2^{-k-3} > 0$. We test both inequalities simultaneously and stop if we have proved one of them to be true. Since both cases are overlapping, this can be done effectively. How we continue depends on the chosen case.

“ $r < 2^{-k-2}$ ”:

In this case the intersection between $B(x, 2^{-k-1})$ and G is not empty and open in G . Since $\delta_{\text{dist}}^{\text{--}} \leq \delta_{\text{range}}$ holds for computable Banach spaces we can effectively find some $g \in G$ such that $\|x - g\| < 2^{-k-1}$, that is $g \in B(x, 2^{-k-1}) \cap G$. Then we have for g

$$\begin{aligned} \|g - g_0\| &\leq \|g - x\| + \|x - g_0\| \\ &= \|g - x\| + r \\ &< 2^{-k-1} + 2^{-k-2} \\ &< 2^{-k} \end{aligned}$$

“ $r > 2^{-k-3}$ ”:

Let $\delta := \min \{2^{-k-1}, 2r \cdot \text{mc}(\frac{1}{r} \cdot 2^{-k-1})\}$. Since mc is computable and we have already computed r , we can compute δ . As we have $\delta > 0$, hence $r + \delta > r$, the intersection between $B(x, r + \delta)$ and G is not empty, and it is open in G . Since $\delta_{\text{dist}}^{\text{--}} \leq \delta_{\text{range}}$ holds for computable Banach spaces we can effectively find some $g \in G$ such that $\|x - g\| < r + \delta$. We prove that $\|g - g_0\| < 2^{-k}$ holds in this case. Therefore,

let $\delta' := \|x - g\| - r < \delta$ and $g' := g - \frac{\delta'}{r+\delta'}(g - x)$. Then we have

$$\|g' - g\| = \frac{\delta'}{r + \delta'} \|g - x\| = \delta' \quad (2)$$

$$\|g' - x\| = \frac{r}{r + \delta'} \|g - x\| = r = \|x - g_0\| \quad (3)$$

$$\left\| \frac{1}{2}(g_0 + g) - x \right\| \geq r \quad (\text{as we have } \frac{1}{2}(g_0 + g) \in G) \quad (4)$$

$$\left\| \frac{1}{2}(g_0 + g) - \frac{1}{2}(g_0 + g') \right\| = \frac{1}{2} \|g - g'\| = \frac{1}{2} \delta' \quad (5)$$

$$\|g_0 - g\| \leq \|g_0 - g'\| + \|g' - g\| = \|g_0 - g'\| + \delta' \quad (6)$$

It follows

$$\begin{aligned} r &\leq \left\| \frac{1}{2}(g_0 + g) - x \right\| \\ &\leq \left\| \frac{1}{2}(g_0 + g) - \frac{1}{2}(g_0 + g') \right\| + \left\| \frac{1}{2}(g_0 + g') - x \right\| \\ &= \left\| \frac{1}{2}(g_0 + g') - x \right\| + \frac{1}{2} \delta' \end{aligned}$$

and furthermore

$$\begin{aligned} \left\| \frac{1}{2}(g_0 + g') - x \right\| &\geq r - \frac{1}{2} \delta' > r - \frac{1}{2} \delta \\ &\geq r - r \cdot \text{mc} \left(\frac{1}{r} \cdot 2^{-k-1} \right) \end{aligned}$$

so that we get

$$\|g_0 - g'\| < 2^{-k-1}$$

Now we can derive the desired result for g

$$\|g_0 - g\| \leq \|g_0 - g'\| + \delta' < \|g_0 - g'\| + \delta < 2^{-k-1} + 2^{-k-1} = 2^{-k}$$

In both cases we can find effectively some $g \in G$ such that $\|g - g_0\| < 2^{-k}$ and our proof is complete. \square

In the next section we show that in case of finite-dimensional linear subspaces, which are particularly nonempty closed convex sets, we can get the same result under less restrictive conditions.

6 Metric Projection onto Linear Subspaces

Given a finite-dimensional linear subspace U of a computable Banach space X and an element x of X , we can compute the set $\mathcal{P}_U(x)$ of all elements of best approximation of x in U as a compact set with negative information. By $\mathcal{K}(X)$ we denote the set of all compact subsets of X and equip $\mathcal{K}(X)$ with the representations $\delta_{\text{min-cover}}$ (full information) and δ_{cover} (only negative information), see also [4].

First, we mention a computability result about metric projections onto linear subspaces that is a simple corollary of the result about nonempty closed convex sets that we have proved in the previous section.

Corollary 6.1 (Metric projection onto subspaces) *Let X be a computable Banach space with a uniformly convex norm and a $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$ -computable modulus of convexity mc . We define a mapping*

$$P_{\mathcal{L}} : \mathcal{L}_X^{(*)} \times X \rightarrow X, (U, x) \mapsto P_U(x)$$

that maps every finite-dimensional linear subspace of X to the corresponding metric projection. Then $P_{\mathcal{L}}$ is $(\delta_{\text{basis}}, \delta_X, \delta_X)$ -computable.

Proof. Every finite-dimensional subspace U of X is a nonempty closed convex subset of X . Since $\delta_{\text{basis}} \leq \delta_{\text{range}}$ and $\delta_{\text{basis}} \leq \delta_{\text{dist}}^{\leq} \leq \delta_{\text{dist}}^{>}$ we have $\delta_{\text{basis}} \leq \delta_{\text{dist}}^{>}$ \square
 δ_{range} . Now we can apply Theorem 5.2. \square

A corresponding result in constructive analysis is Theorem 3.1 in [6].

Since finite-dimensional linear subspaces are very special nonempty closed convex subsets, it arises the question if we can for example get a result without demanding a computable modulus of convexity. In fact, this is possible and we can get some better computability results about metric projections in this case. Given a finite-dimensional linear subspace U of a Banach space X and an element x of X , we can compute the set $\mathcal{P}_U(x)$ of all best approximations of x in U as a compact set with negative information.⁷

Theorem 6.2 (Metric projection onto subspaces) *Let X be a computable Banach space. We define a mapping*

$$P_{\mathcal{L}}^{\text{set}} : \mathcal{L}_X^{(*)} \times X \rightarrow \mathcal{K}(X), (U, x) \mapsto \mathcal{P}_U(x)$$

that maps every finite-dimensional linear subspace U and element x to the corresponding compact set of elements of best approximation. Then the mapping $P_{\mathcal{L}}^{\text{set}}$ is $(\delta_{\text{dist}}^{\leq}, \delta_X, \delta_{\text{cover}})$ -computable and also $(\delta_{\text{basis}}, \delta_X, \delta_{\text{cover}})$ -computable.

Proof. Given a δ_{basis} -name of a finite-dimensional subspace U of X , we can compute a $\delta_{\text{dist}}^{\leq}$ -name of U . Thus it suffices to prove the $(\delta_{\text{dist}}^{\leq}, \delta_X, \delta_{\text{cover}})$ -computability. Using the given $\delta_{\text{dist}}^{\leq}$ -name of U in combination with the given $x \in X$, we can compute a $[\delta_X \rightarrow \delta_{\mathbb{R}}]$ -name of the function $f : X \rightarrow \mathbb{R}$ defined by $f(z) := |||x - z|| - \text{dist}(x, U)|$ for $z \in X$. It holds $f(z) = 0 \iff z \in S(x, \text{dist}(x, U))$ for all $z \in X$. Hence f represents a δ_{fiber} -name of the sphere with center x and radius $\text{dist}(x, U)$. Since every element of best approximation of x in U has got the distance $\text{dist}(x, U)$ from U it holds $\mathcal{P}_U(x) = U \cap S(x, \text{dist}(x, U)) \subseteq S(x, \text{dist}(x, U))$. We have $\text{dist}(x, U) \leq ||x||$ because of $0 \in U$. Thus we have $\mathcal{P}_U(x) \subseteq B_X(0, 2||x||) \cap U = B_U(0, 2||x||)$. Since $B_U(0, 2||x||)$ is a finite-dimensional compact subset of X , we can

⁷ Since the set of all best approximations is closed and bounded, it is compact in case of finite-dimensional subspaces.

compute a $\delta_{\min\text{-cover}}$ -name of $B_U(0, 2\|x\|)$. Given a closed set with negative information and a compact set with negative information, we can compute the intersection of these two sets with negative information. Thus we can compute a δ_{cover} -name of $P_U(x) = S(x, \text{dist}(x, U)) \cap B_U(0, 2\|x\|)$. \square

By some results of Brattka [3,2] it is possible to compute the unique element x of the set $\{x\}$ assumed that we have got negative information of $\{x\}$ as a compact set. This fact leads us to the following result.

Theorem 6.3 (Metric projection) *Let X be a computable Banach space with a uniformly convex norm. We define a mapping*

$$P_{\mathcal{L}} : \mathcal{L}_X^{(*)} \times X \rightarrow X, (U, x) \mapsto P_U(x)$$

that maps every finite-dimensional linear subspace of X to the corresponding metric projection. Then $P_{\mathcal{L}}$ is $(\delta_{\text{dist}}^{\text{=}}, \delta_X, \delta_X)$ -computable and also $(\delta_{\text{basis}}, \delta_X, \delta_X)$ -computable.

Proof. By Theorem 6.2 given a finite-dimensional subset U and an element x we can compute a δ_{cover} -name of $\mathcal{P}_U(x)$. If X is uniformly convex $\mathcal{P}_U(x)$ consists of the single point $P_U(x)$. We can compute a δ_X -name of $P_U(x)$ from the δ_{cover} -name of $\mathcal{P}_U(x) = \{P_U(x)\}$ as it is possible to convert the negative information of a singleton as compact set to a name of its unique element [3,2]. \square

Thus, in the special case of finite-dimensional subspaces we do not need the computable modulus of convexity that we used in the more general case of convex subsets in Theorem 5.2.

Given a computable Banach space X , we can also define a (partial) mapping

$$P_{\mathcal{L}}^{\text{unique}} : \subseteq \mathcal{L}_X^{(*)} \times X \rightarrow X, (U, x) \mapsto P_U(x)$$

with

$$\text{dom}(P_{\mathcal{L}}^{\text{unique}}) := \{(U, x) \in \mathcal{L}_X^{(*)} \times X \mid |\mathcal{P}_U(x)| = 1\}$$

that maps every finite-dimensional linear subspace U and element x with a unique element of best approximation in U to its best approximation. Using Theorem 6.2 and the already used results from [3,2], we obtain that the mapping $P_{\mathcal{L}}^{\text{unique}}$ is $(\delta_{\text{dist}}^{\text{=}}, \delta_X, \delta_X)$ -computable and $(\delta_{\text{basis}}, \delta_X, \delta_X)$ -computable. It seems that this result could also be derived from [7, Theorem 1] (also [1, Chapter 7, Theorem 2.12] and [5, Theorem 4.2.1]) via realizability theory.

7 Conclusions

In this paper we have studied the computability of the metric projection onto closed convex sets and finite-dimensional linear subspaces of computable Banach spaces X , which is equivalent to computing the best approximation of an element in a given subset. In uniformly convex Banach spaces with computable modulus of convexity

we can compute the (unique) best approximation of a given element of X in an also given nonempty closed convex subset of X . In case of finite-dimensional linear subspaces of Banach spaces, which are special closed convex subsets, we can omit the requirement of a computable modulus of convexity. Without the condition of uniform convexity we can compute the set of best approximations as a compact subset. If we additionally assume uniform convexity we can compute the unique element of this compact subset, that is the metric projection.

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